Homework 3: Solutions to exercises not appearing in Pressley, also 2.2.1, 2.2.3, 2.2.5, 2.2.8, 2.2.9

Math 120A

- (2.1.4) Let $\gamma(t) = (\sec t, \sec t \tan t, 0)$ on $\frac{-\pi}{2} < t < \frac{\pi}{2}$. Then

$$\begin{aligned}
\dot{\gamma}(t) &= (\sec t \tan t, \sec t \tan^2 t + \sec^3 t, 0) \\
\ddot{\gamma}(t) &= (\sec t \tan^2 t + \sec^3 t, \sec t \tan^3 t + 2\sec^3 t \tan t + 3\sec^3 t \tan t, 0) \\
&= \sec t(\tan^2 t + \sec^2 t, \tan^3 t + 5\sec^2 t \tan t, 0)
\end{aligned}$$

To see where the curvature vanishes, it suffices to determine for what values of t $\ddot{\gamma}(t) \times \dot{\gamma}(t) = 0$. This vector is $(0, 0, \lambda)$, with λ as follows:

$$\begin{aligned}
\lambda &= \sec^2 t((\tan^2 t + \sec^2 t)^2 - \tan t(\tan^3 t + 5\sec^2 t \tan^2 t)) \\
&= \sec^2 t(\tan^4 t + 2\tan^2 t \sec^2 t + \sec^4 t - \tan^4 t - 5\sec^2 t \tan^2 t) \\
&= \sec^2 t(\sec^4 t - 3\tan^2 t \sec^2 t) \\
&= \sec^4 t(\sec^2 t - 3\tan^2 t) \\
&= \sec^4((\sec^2 t - \tan^2 t) - 2\tan^2 t) \\
&= \sec^4 t(1 - 2\tan^2 t)
\end{aligned}$$

Since $\sec t$ is nonzero on $\frac{-\pi}{2} < t < \frac{\pi}{2}$, the curvature can only be zero when $1 = 2\tan^2 t$, or at $t = \pm \arctan\left(\frac{1}{\sqrt{2}}\right) = \pm \arcsin\left(\frac{1}{\sqrt{3}}\right)$.

- (2.2.3) Let $M(\mathbf{x}) = Q\mathbf{x} + \mathbf{a}$ be an isometry of the plane. Then if $\gamma(s)$ is a unit-speed curve, let $\Gamma = M \circ \gamma$. Now $\dot{\Gamma}(s) = Q\dot{\gamma}(s)$ and $\ddot{\Gamma}(s) = Q\ddot{\gamma}(s)$. Therefore since multiplication by Q is length-preserving (see the problem below on isometries), $||\ddot{\Gamma}(s)|| = ||Q\ddot{\gamma}(s)||$, and the two curves have the same curvature. If Q is a rotation, it's clear $\mathbf{N}_s = Q\mathbf{n}_s$, so the signed curvatures of Γ and γ agree. If Q is a reflection across the y-axis, $\mathbf{N}_s = -Q\mathbf{n}_s$, so the signed curvatures of Γ and γ are opposite. Since every isometry of the plane is a composition of rotation, reflection across the y-axis, and translation, we are done.

  For the converse, if γ and $\widetilde{\gamma}$ have the same nonzero curvature functions, either their signed curvatures are the same, in which case they are related by a direct isometry (as proved in class), or they differ by a factor of -1. In the second case we can reflect γ across the y-axis and obtain a curve with the same signed curvature as $\widetilde{\gamma}$ which is related by direct isometry to $\widetilde{\gamma}$. Ergo γ and $\widetilde{\gamma}$ are related by opposite isometry.

- (2.2.1) Recall that $\mathbf{t} \perp \mathbf{n}_s$, and $\dot{\mathbf{t}} = \kappa_s \mathbf{n}_s$. So $\mathbf{t} \cdot \mathbf{n}_s = 0$. Differentiating this relationship gives

$$\begin{aligned}
\dot{\mathbf{t}} \cdot \mathbf{n}_s + \mathbf{t} \cdot \dot{\mathbf{n}}_s &= 0 \\
\kappa_s \mathbf{n}_s \cdot \mathbf{n}_s + \mathbf{t} \cdot \mathbf{n}_s &= 0 \\
\mathbf{t} \cdot \mathbf{n}_s &= -\kappa_s
\end{aligned}$$

Because that $\mathbf{n}_s$ is a unit vector, $\mathbf{n}_s \perp \dot{\mathbf{n}}_s$, so $\mathbf{n}_s$ is colinear with $\mathbf{t}$. We conclude that $\mathbf{n}_s = -\kappa_s \mathbf{t}$.

- (2.2.5) We have $\gamma(t)$ regular and $\gamma^\lambda(t) = \gamma(t) + \lambda \mathbf{n}_s(t)$. We see that

$$\begin{aligned}\dot{\gamma}^\lambda(t) &= \dot{\gamma}(t) + \lambda \frac{d\mathbf{n}_s}{dt} \\ &= ||\dot{\gamma}(t)||\mathbf{t} - \lambda \frac{d\mathbf{n}_s}{ds}\frac{ds}{dt} \\ &= ||\dot{\gamma}(t)||\mathbf{t} - \lambda\kappa_s \mathbf{t}||\dot{\gamma}(t)|| \\ &= (1-\lambda\kappa_s)||\dot{\gamma}(t)||\mathbf{t}.\end{aligned}$$

We conclude that whenever $\kappa_s\lambda \neq 1$, $\gamma^\lambda(t)$ is regular with $\frac{ds^\lambda}{dt} = |1-\kappa_s\lambda|||\dot{\gamma}(t)||$. Now we can discuss curvature. The unit tangent vector $\mathbf{t}^\lambda = \frac{\dot{\gamma}^\lambda(t)}{||\dot{\gamma}^\lambda(t)||} = \pm\mathbf{t}$ according to whether $1-\lambda\kappa_s$ is positive or negative. That means $\mathbf{n}_s^\lambda = \pm\mathbf{n}_s$ with the same sign. Now we differentiate $\mathbf{t}^\lambda$ with respect to its arclength s^λ, obtaining

$$\begin{aligned}\frac{d\mathbf{t}^\lambda}{ds^\lambda} &= \frac{d\mathbf{t}^\lambda}{ds}\frac{ds}{ds^\lambda} \\ &= \pm\frac{d\mathbf{t}}{ds}\frac{1}{|1-\lambda\kappa_s|} \\ &= \pm\kappa_s\mathbf{n}_s\frac{1}{|1-\lambda\kappa_s|} \\ &= \frac{\kappa_s}{|1-\lambda\kappa_s|}(\pm\mathbf{n}_s)\end{aligned}$$

Since $\pm\mathbf{n}_s$ is the signed unit normal of $\gamma^\lambda(t)$, we conclude that the signed curvature is $\frac{\kappa_s}{|1-\lambda\kappa_s|}$.

- (2.2.8) We have $\iota(s) = \gamma(s) + (\ell - s)\dot{\gamma}(s)$, and $\kappa_s \neq 0$. Now

$$\begin{aligned}\dot{\iota}(s) &= \dot{\gamma}(s) - \dot{\gamma}(s) + (\ell - s)\ddot{\gamma}(s) \\ &= (\ell - s)\ddot{\gamma}(s) \\ &= (\ell - s)\kappa_s\mathbf{n}_s\end{aligned}$$

Therefore if v is the arclength parameter for the involute, we see that $\frac{dv}{ds} = (\ell - s)\kappa_s$. Moreover, the unit tangent vector of the involute is $\mathbf{n}_s$, which after rotation implies that the signed unit normal is $-\mathbf{t}$. Ergo to find the signed curvature of ι, we should differentiate $\mathbf{n}_s$ with respect to $\mathbf{v}$, as follows.

$$\begin{aligned}\frac{d\mathbf{n}_s}{dv} &= \frac{d\mathbf{n}_s}{ds}\frac{ds}{dv} \\ &= -\kappa_s\mathbf{t}\frac{1}{(\ell - s)\kappa_s} \\ &= \frac{1}{\ell - s}(-\mathbf{t})\end{aligned}$$

Ergo the signed curvature of the involute is $\frac{1}{\ell - s}$.

- (2.2.9) We have $\gamma(t) = (t, \cosh t)$. To find its involute, the first thing to do is reparametrize with respect to the arc length. We see that $\dot{\gamma}(t) = (1, \sinh t)$, so the arclength is

$$\begin{aligned} s &= \int_0^t \sqrt{1+\sinh^2(u)}du \\ &= \int_0^t \cosh u du \\ &= \sinh t \end{aligned}$$

Ergo the arclength reparametrization is $\gamma(s) = (\sinh^{-1}(s), \sqrt{1+s^2})$. Therefore the involute is

$$\begin{aligned} \iota(s) &= \left(\sinh^{-1}(s), \sqrt{1+s^2}\right) - s\left(\frac{1}{\sqrt{s^2+1}}, \frac{s^2}{\sqrt{s^2+1}}\right) \\ &= \left(\sinh^{-1}(s) - \frac{s}{\sqrt{s^2+1}}, \frac{1}{\sqrt{s^2+1}}\right) \\ &= \left(u - \frac{\sinh u}{\cosh u}, \frac{1}{\cosh u}\right) \\ &= (u - \tanh u, \operatorname{sech} u) \end{aligned}$$

Here the second-to-last step makes the substitution $u = \sinh^{-1} s$ to simplify the equations. Now we have $x = u - \tanh u$ and $y = \operatorname{sech} u$. Therefore from the second equation, $u = \cosh^{-1}\left(\frac{1}{y}\right)$, so $x = \cosh^{-1}\left(\frac{1}{y}\right) - \sqrt{1-y^2}$, since $1 - \operatorname{sech}^2 u = \tanh^2 u$.

- Question 3: Isometries. (a) Suppose $M(\mathbf{0}) = \mathbf{0}$ and $M(\mathbf{e}_i) = \mathbf{e}_i$ for any i. Then for any $\mathbf{v} \in \mathbb{R}^n$, $||M(\mathbf{v}) - 0||^2 = ||\mathbf{v} - 0||^2$, implying that $M(\mathbf{v}) \cdot M(\mathbf{v}) = \mathbf{v}\dot{\mathbf{v}}$. Moreover, $||M(\mathbf{v}) - e_i||^2 = ||\mathbf{v} - e_i||^2$. Expanding this equation we see

$$\begin{aligned} M(\mathbf{v}) \cdot M(\mathbf{v}) - 2M(\mathbf{v}) \cdot \mathbf{e}_i + \mathbf{e}_i \cdot \mathbf{e}_i &= \mathbf{v} \cdot \mathbf{v} - 2\mathbf{v} \cdot \mathbf{e}_i + \mathbf{e}_i \cdot \mathbf{e}_i \\ M(\mathbf{v}) \cdot M(\mathbf{v}) - 2M(\mathbf{v}) \cdot \mathbf{e}_i &= \mathbf{v} \cdot \mathbf{v} - 2\mathbf{v} \cdot \mathbf{e}_i \\ M(\mathbf{v}) \cdot \mathbf{e}_i &= \mathbf{v} \cdot \mathbf{e}_i \end{aligned}$$

This shows that the ith coordinates of $\mathbf{v}$ and $\mathbf{v}$ are the same for all i, so $M(\mathbf{v}) = \mathbf{v}$.

(b) If $M(0) = 0$ and $M(\mathbf{e}_i) = \mathbf{v}_i$, then $||\mathbf{v}_i - 0|| = ||\mathbf{e}_i - 0||$, so $\mathbf{v}_i$ is a unit vector. Moreover, we expand as in the previous part, obtaining

$$\begin{aligned} ||\mathbf{v}_i - \mathbf{v}_j||^2 &= ||\mathbf{e}_i - \mathbf{e}_j||^2 \\ \mathbf{v}_i \cdot \mathbf{v_i} - 2\mathbf{v}_i \cdot \mathbf{v}_j + \mathbf{v}_j \cdot \mathbf{v}_j &= \mathbf{e}_i \cdot \mathbf{e}_i - 2\mathbf{e}_i \cdot \mathbf{e}_j + \mathbf{e}_j \cdot \mathbf{e}_j \\ 2 - 2\mathbf{v}_i \cdot \mathbf{v}_j &= 2 \\ \mathbf{v}_i \cdot \mathbf{v}_j &= 0 \end{aligned}$$

Therefore the $\mathbf{v}_i$ form an orthonormal system. However, we certainly know of an isometry that takes each $\mathbf{e}_i$ to $\mathbf{v}_i$, namely multiplication by the matrix Q whose ith column is

$\mathbf{v}_i$. (To see this is an isometry, observe that it preserves all lengths: if $\mathbf{c} = (c_1, \cdots, c_n)$, then $||Q\mathbf{c}||^2 = (c_1\mathbf{v}_1 + \cdots + c_n\mathbf{v}_n) \cdot (c_1\mathbf{v}_1 + \cdots + c_n\mathbf{v}_n) = c_1^2 + \cdots c - n^2 = ||\mathbf{c}||^2$.) We observe that $Q^{-1} \circ M$ is an isometry fixing $\mathbf{0}$ and each $\mathbf{e}_i$. By part (a), this means $Q^{-1} \circ M$ is the identity map, implying that M is just multiplication by $\mathbf{Q}$.

(c) Now let M be an arbitrary isometry. Let $\mathbf{a} = M(\mathbf{0})$. Then let $N = T_{-\mathbf{a}} \circ M$, so that N is an isometry with $N(\mathbf{0}) = \mathbf{0}$. By part (b), N is equal to multiplication by an orthogonal matrix Q. Then since $N = Q\mathbf{x}$, we see that $M = T_{\mathbf{a}} \circ N$ is given by $M(x) = Q\mathbf{x} + \mathbf{a}$. We have already seen in part (b) that multiplication by an orthogonal matrix is distance-preserving, and translation clearly also is, so the converse follows.

(d) Suppose that Q is an orthogonal matrix in $\mathbb{R}^2$, where Q is

$$\begin{pmatrix} a & c \\ b & d \end{pmatrix}.$$

Since $QQ^t = I$, we see that $a^2 + b^2 = 1$, $c^2 + d^2 = 1$, and $ad - bc = 0$. We can always find θ such that $a = \cos\theta$ and $b = \sin\theta$. Then our remaining choices are $c = -\sin\theta$, $d = \cos\theta$, which gives a rotation counterclockwise by θ, and $c = \sin\theta$, $d = \cos\theta$, which gives a reflection across the y-axis followed by a rotation counterclockwise by θ, or equivalently a reflection through the line $\theta = \frac{\pi}{2}$.